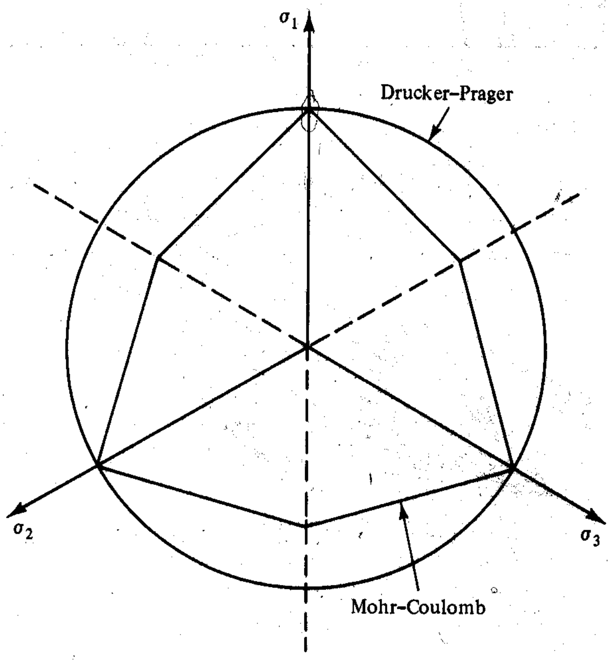


Geomechanics

LECTURE 4

PLASTICITY

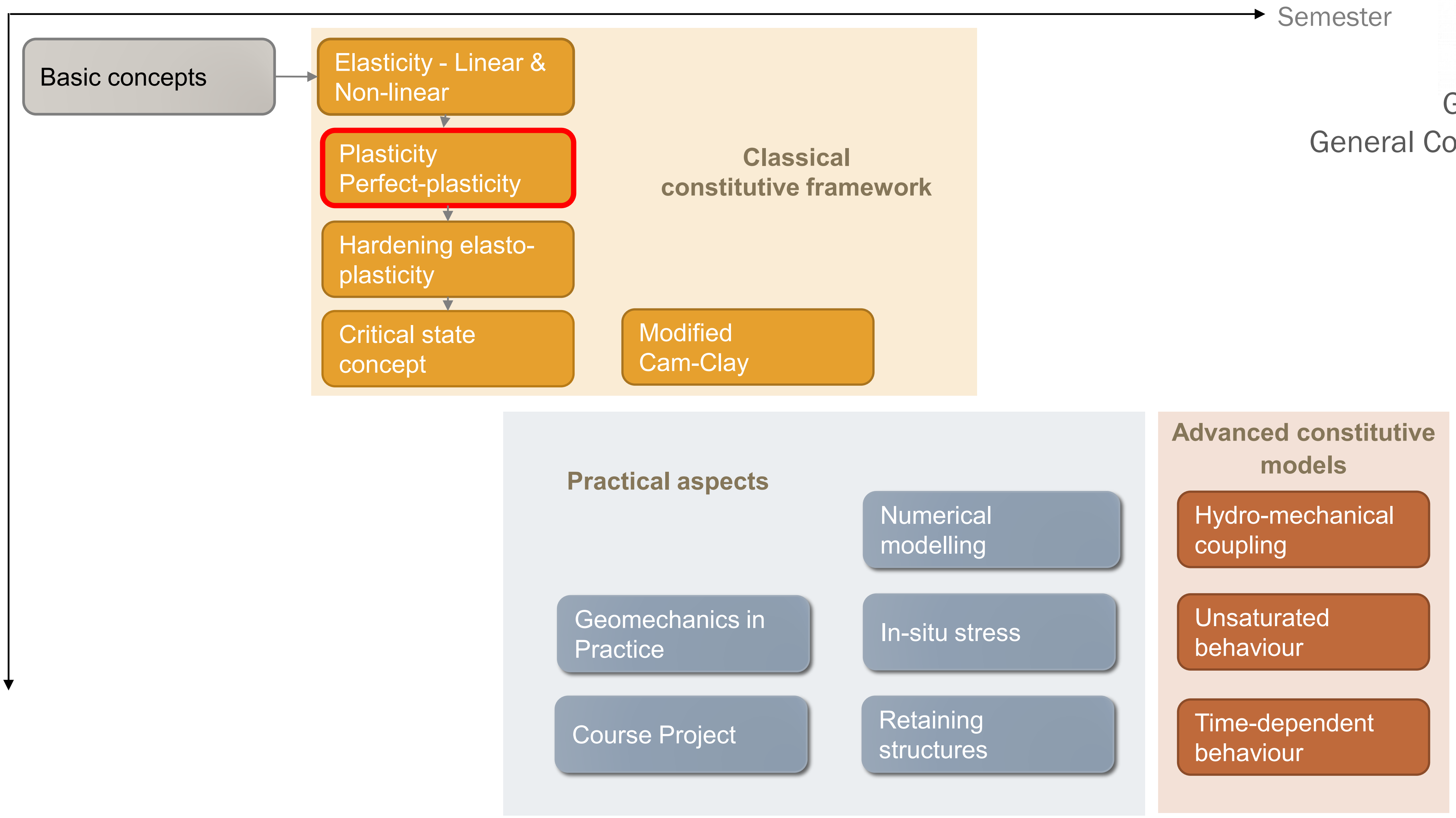
Dr. ALESSIO FERRARI

Laboratory of soil mechanics – Fall 2025

Access the QUIZ

https://etc.ch/fTyf

Geomechanics
General Course overview



Semester

Basic concepts

Classical constitutive framework

Elasticity - Linear & Non-linear

Plasticity Perfect-plasticity

Hardening elasto-plasticity

Critical state concept

Modified Cam-Clay

Practical aspects

Numerical modelling

Geomechanics in Practice

In-situ stress

Course Project

Retaining structures

Advanced constitutive models

Hydro-mechanical coupling

Unsaturated behaviour

Time-dependent behaviour

Content

- Plasticity principle & failure criteria
- Elastic-perfectly plastic models:
  - Von Mises
  - Drucker-Prager
  - Mohr-Coulomb
- Application of elastic-perfectly plastic models
- Conclusion

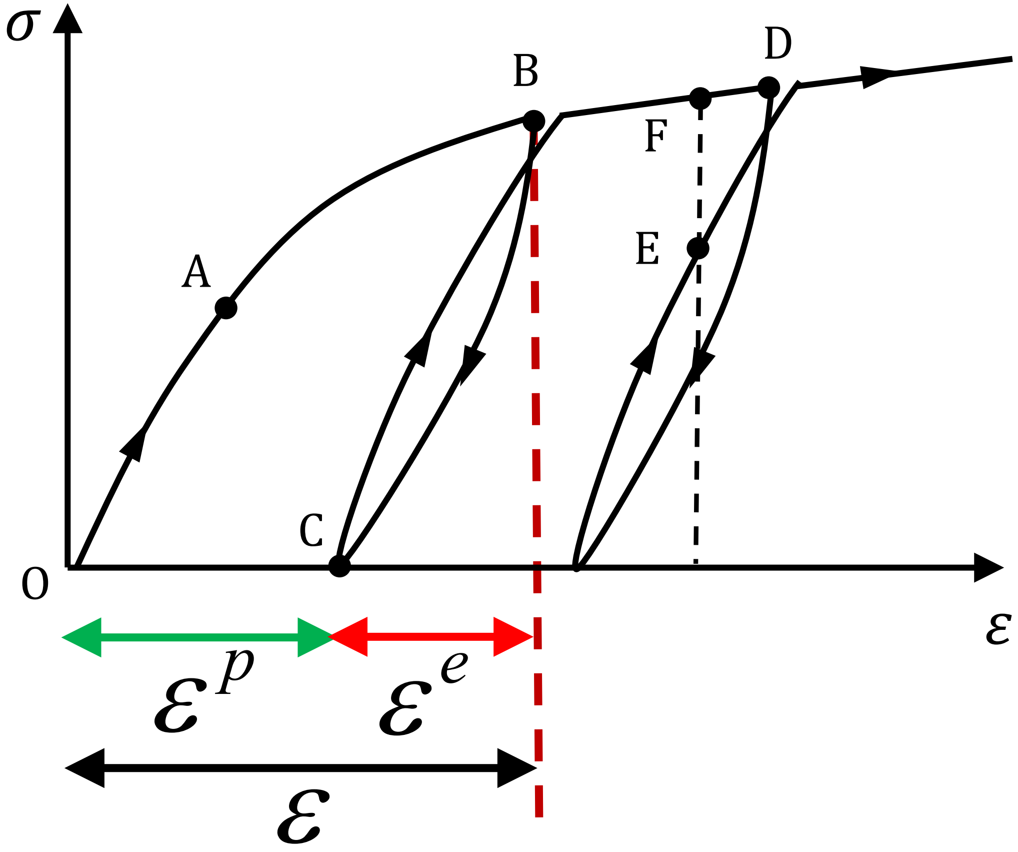
Plasticity principle & failure criteria

PLASTICITY

YIELD CRITERIA

Plasticity – generalities

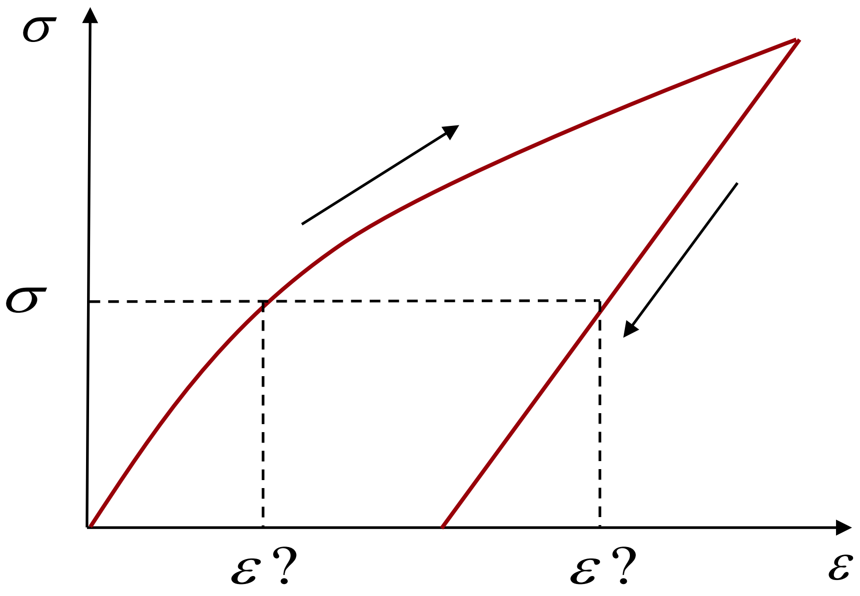
- Plasticity - simple definition: irreversible deformation

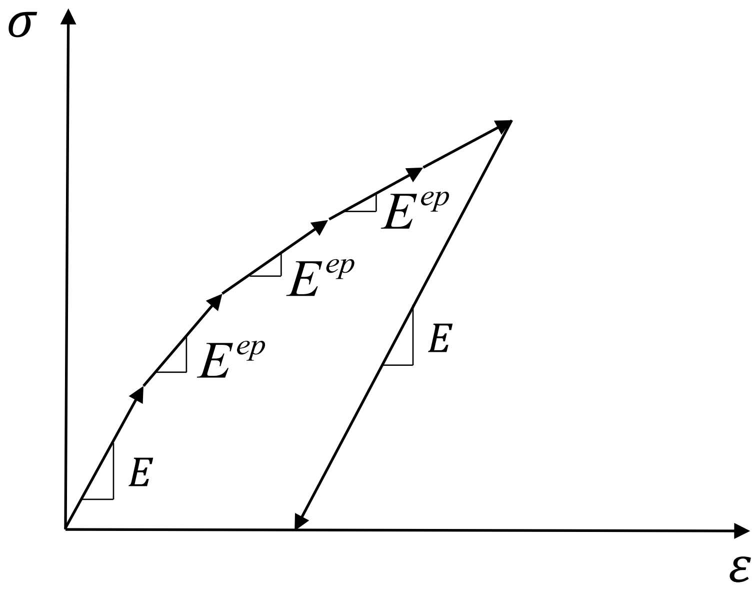


$$\varepsilon = \varepsilon^e + \varepsilon^p$$

Plasticity – generalities

- Strain state depends not only on the actual stress state but also on the stress history
- Need to refer to incremental loading

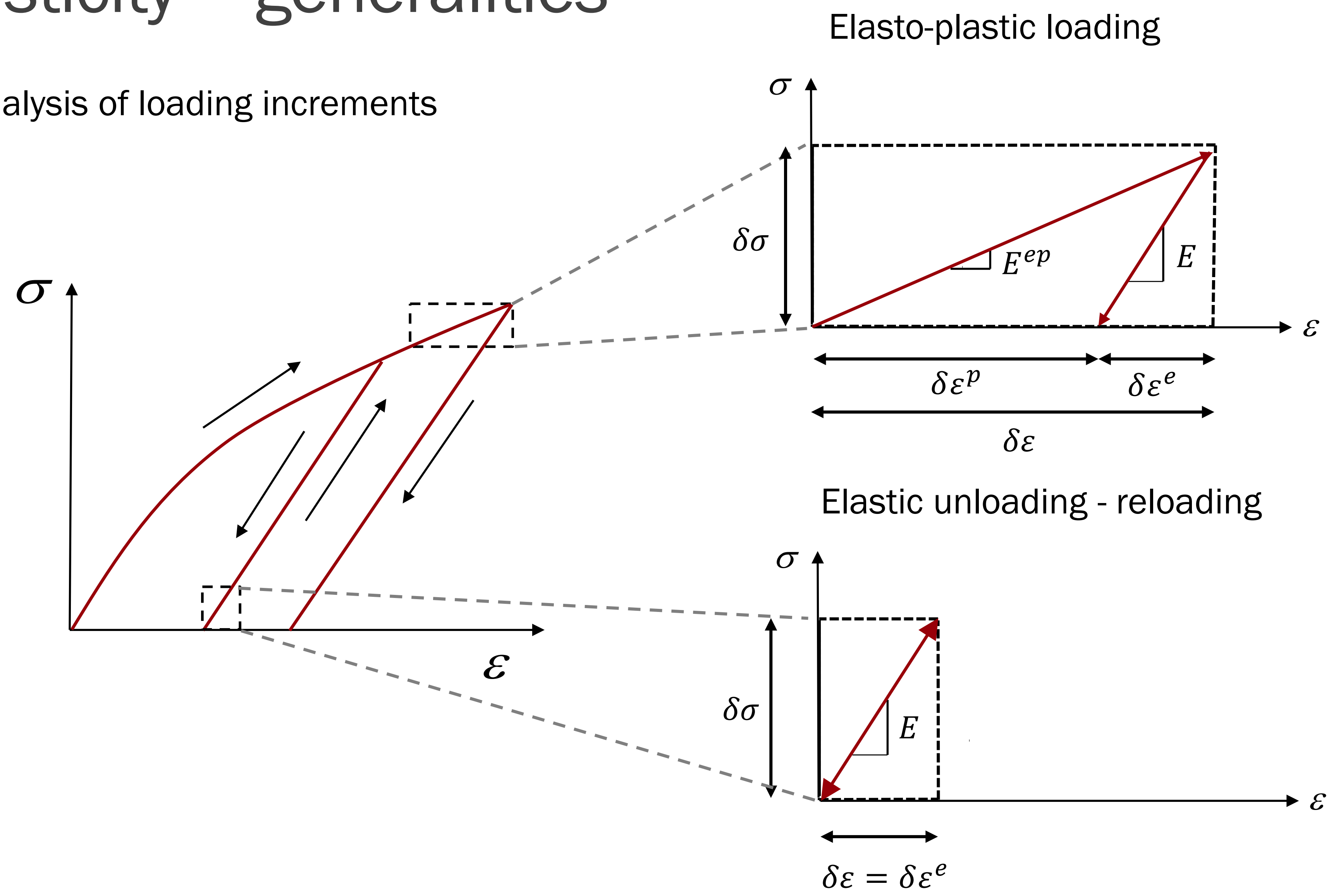




E^{ep} different for each increment

Plasticity – generalities

- Analysis of loading increments



Elasto-plastic loading

Elastic unloading - reloading

$$\delta\sigma = E^{ep}\delta\varepsilon$$

$$\delta\varepsilon = \delta\varepsilon^e + \delta\varepsilon^p$$

$$\delta\varepsilon^e = \frac{\delta\sigma}{E}$$

$$\varepsilon^e = \frac{\sigma}{E}$$

$$\delta\varepsilon = \delta\varepsilon^e$$

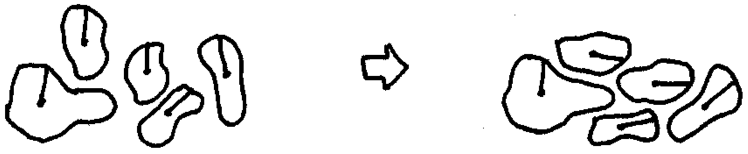
$$\delta\sigma = E\delta\varepsilon^e$$

$$\delta\varepsilon^e = \frac{\delta\sigma}{E}$$

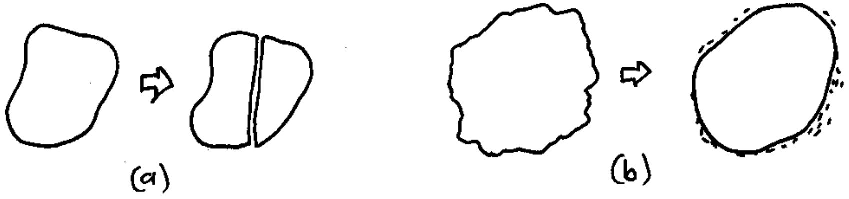
$$\varepsilon^e = \frac{\sigma}{E}$$

Physical causes of plastic deformation

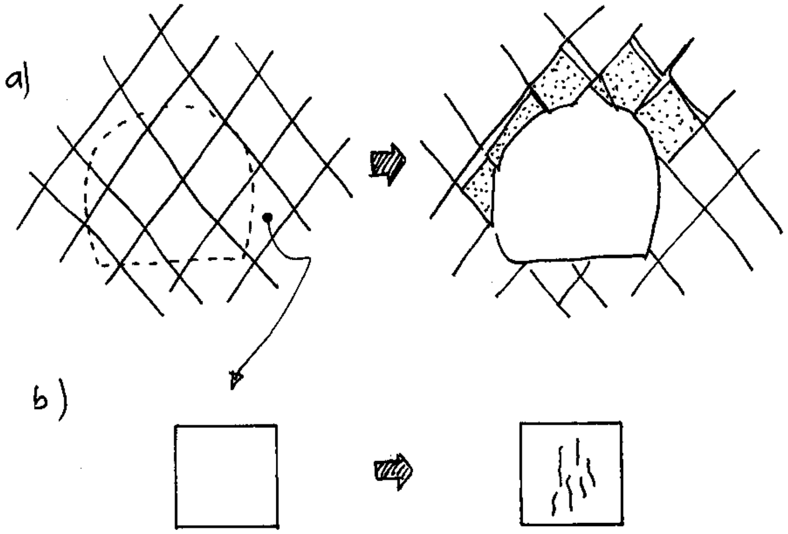
Particle rearrangement



Particle crushing



Sliding along the joints or micro-fracturing (rock)



Basic ingredients

1. Decomposition of strains:

$$\delta\varepsilon_{ij} = \delta\varepsilon_{ij}^{e} + \delta\varepsilon_{ij}^{p} \qquad \delta\varepsilon_{i} = \delta\varepsilon_{i}^{e} + \delta\varepsilon_{i}^{p}$$

2. The elastic constitutive tensor/matrix needs to be defined.

Direct link between effective stress and elastic strain:

$$\sigma'_{ij} = D_{ijhk}^{e}\varepsilon_{hk}^{e} \qquad \sigma'_{i} = D_{ij}^{e}\varepsilon_{j}^{e}$$

In incremental form:

$$\delta\sigma'_{ij} = D_{ijhk}^{e}\delta\varepsilon_{hk}^{e} \qquad \delta\sigma'_{i} = D_{ij}^{e}\delta\varepsilon_{j}^{e}$$

Basic ingredients

3. Definition of the yield function: a set of mathematical condition for yielding

$$\boxed{F(\sigma_{ij}, p_k)}$$

p_k is a collection of parameters

In terms of stress components $F = F(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}, p_k)$

In terms of principal stresses (isotropic material) $F = F(\sigma_1, \sigma_2, \sigma_3, p_k)$

In terms of triaxial stress variables $F = F(p', q, p_k)$

Basic ingredients

The yield function becomes a yield surface:

$$\boxed{F\left(\sigma_{ij}, p_k\right) = 0}$$

which is used to express the limit of the elastic region.

F must be defined in order to ensure the following conditions:

$F\left(\sigma_{ij}, p_k\right) < 0$ if the stress point stays inside the surface

$F\left(\sigma_{ij}, p_k\right) = 0$ if the stress point stays on the surface

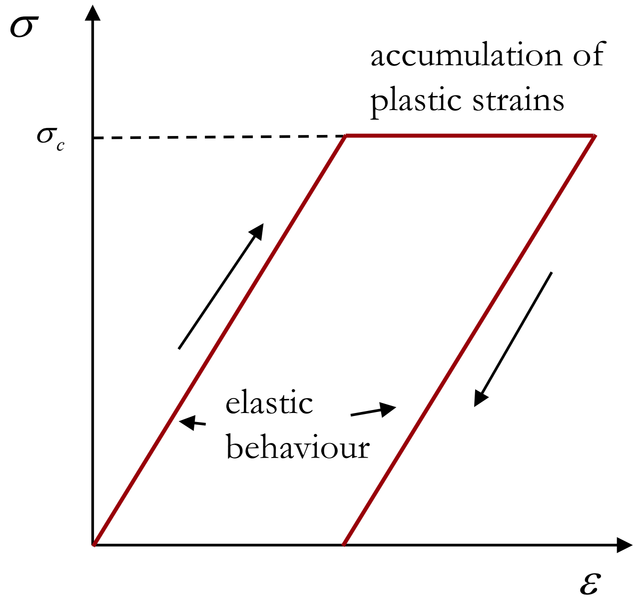
$F\left(\sigma_{ij}, p_k\right) > 0$ expresses an impossible condition

Basic ingredients

Example of a yield function/surface in 1D stress state

A single stress component is needed (σ). Yielding occurs when the stress reaches the maximum allowed value (σ_c), which becomes the geometrical parameter (p_k).

Yield function	Yield surface
$F(\sigma_{ij}, p_k)$	$F(\sigma_{ij}, p_k) = 0$
$F = \sigma - \sigma_c$	$F = \sigma - \sigma_c = 0$



This definition ensures the respect of the conditions stated previously, in fact:

- If $\sigma < \sigma_c \rightarrow F < 0$ elastic behaviour
- If $\sigma = \sigma_c \rightarrow F = 0$ now accumulation of plastic strain
- If $\sigma > \sigma_c \rightarrow F > 0$ impossible

Basic ingredients

Example of a yield function/surface in 2D stress state

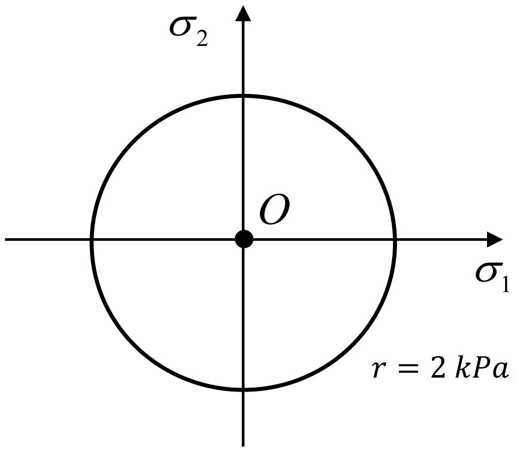
Two stress components are considered (for example the principal stresses σ_1, σ_2). Yielding occurs when the stress point reaches the limit of the elastic domain (for example a circle of radius $r = 2$ kPa, centered in the origin of the σ_1, σ_2 plane).

Yield function $\quad F(\sigma_{ij}, p_k) \rightarrow F(\sigma_1, \sigma_2, r) = (\sigma_1)^2 + (\sigma_2)^2 - r^2 \rightarrow F(\sigma_1, \sigma_2, 2) = (\sigma_1)^2 + (\sigma_2)^2 - 4$

Yield surface $\quad F(\sigma_{ij}, p_k) = 0 \rightarrow F(\sigma_1, \sigma_2, r) = 0 \rightarrow (\sigma_1)^2 + (\sigma_2)^2 - r^2 = 0 \rightarrow (\sigma_1)^2 + (\sigma_2)^2 - 4 = 0$

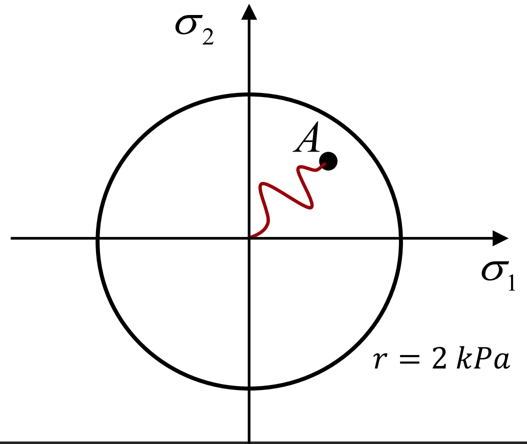
- In the center point O:

$F = -4 < 0$ elastic behaviour



- For each point of the red path:

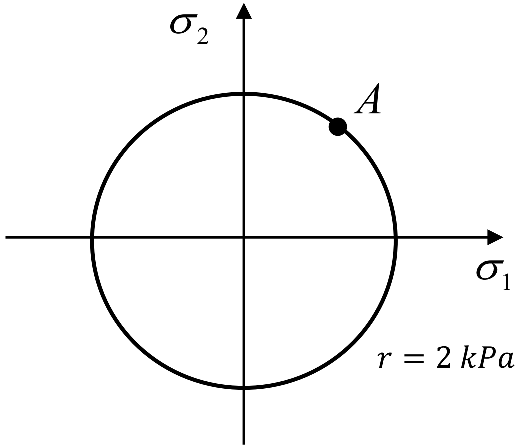
$F < 0$ elastic behaviour



- For a point on the surface $F = 0$

For example in point A ($\sigma_1 = \sigma_2 = \sqrt{2}\ kPa$):

$F(A) = (\sqrt{2})^2 + (\sqrt{2})^2 - 4 = 0$ plastic strain accumulation



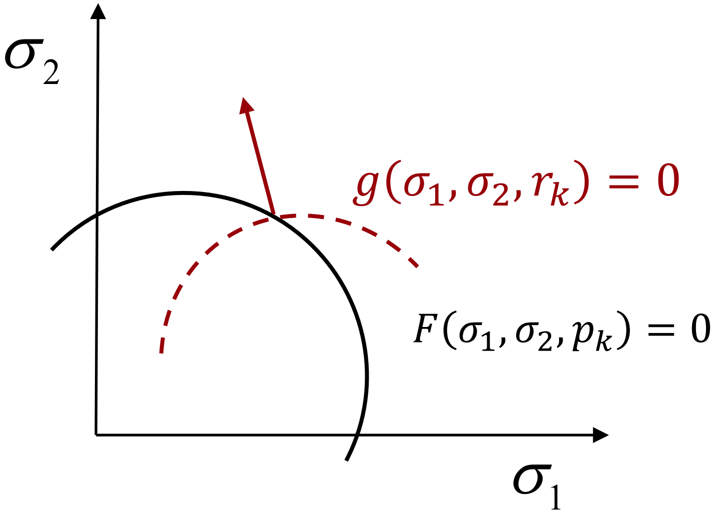
Basic ingredients

4. Definition of the plastic potential $g(\sigma_{ij}, r_k)$

r_k is a collection of geometrical parameters

g is defined so that the components of plastic strain increments can be computed as:

$$\delta\varepsilon_{ij}^{p} = \mu\frac{\partial g}{\partial \sigma_{ij}} \qquad \delta\varepsilon_{i}^{p} = \mu\frac{\partial g}{\partial \sigma_{i}}$$



This expression is called the *flow rule*

The plastic deformation vector has the direction of the gradient of the surface $g = 0$.

μ (or often called λ) is the plastic multiplier and it will become the unknown in the elasto-plastic formulation to be solved (see consistency equation after).

If we assume $g = F$ the plasticity is called associated plasticity.

Consistency Condition

For a point on F:

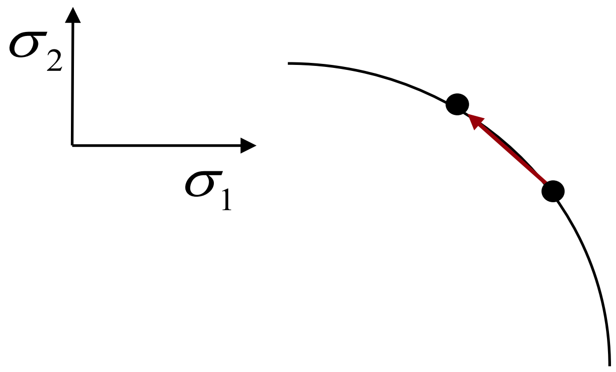
$$F(\sigma_i, p_k) = 0$$

When the behaviour is not elastic, also the final position of the increment must stay on F:

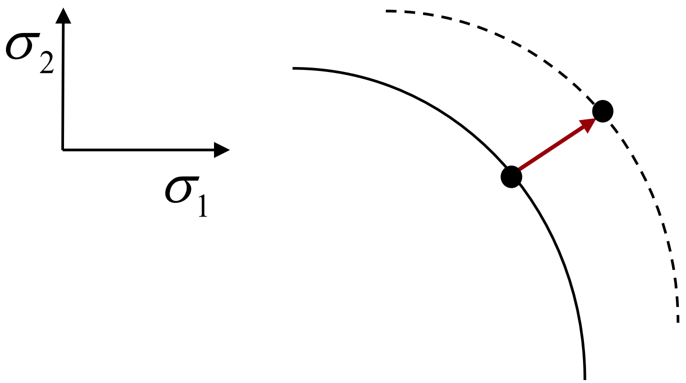
$$F(\sigma_i + \delta\sigma_i, p_k + \delta p_k) = 0$$

Two possibilities:

1. Stress path stays on F
F doesn't evolve: no change in p_k



2. Stress path is allowed to exceed F
F has to evolve to accommodate the final position of the stress path: changes in p_k



Consistency Condition

In both cases (1) and (2):

$$F(\sigma_i, p_k) = 0$$

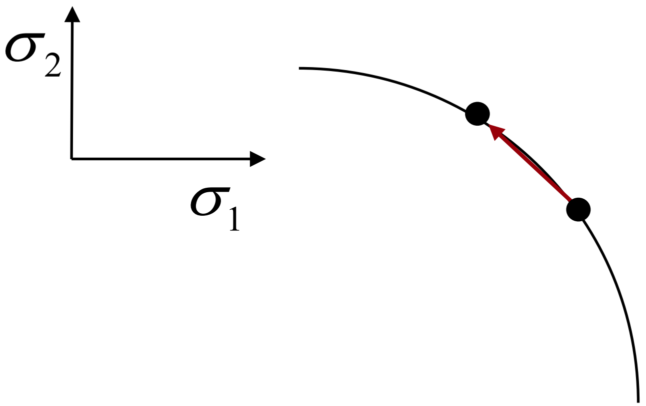
And:

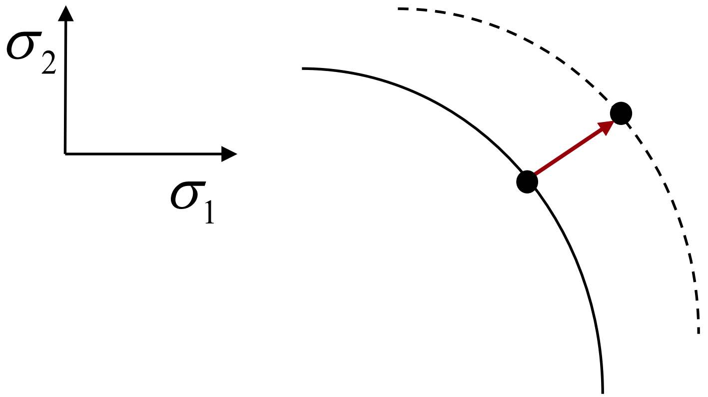
$$F(\sigma_i + \delta\sigma_i, p_k + \delta p_k) = 0$$

Hence:

$$dF = \left.\frac{\partial F}{\partial \sigma_i}\right|_{p_k} \delta\sigma_i + \left.\frac{\partial F}{\partial p_k}\right|_{\sigma_i} \delta p_k = 0$$

- $\frac{\partial F}{\partial \sigma_i}$ is the gradient of F;
- $\frac{\partial F}{\partial \sigma_i}\delta\sigma_i$ is a scalar product.





Elastic-perfectly plastic models

VON MISES

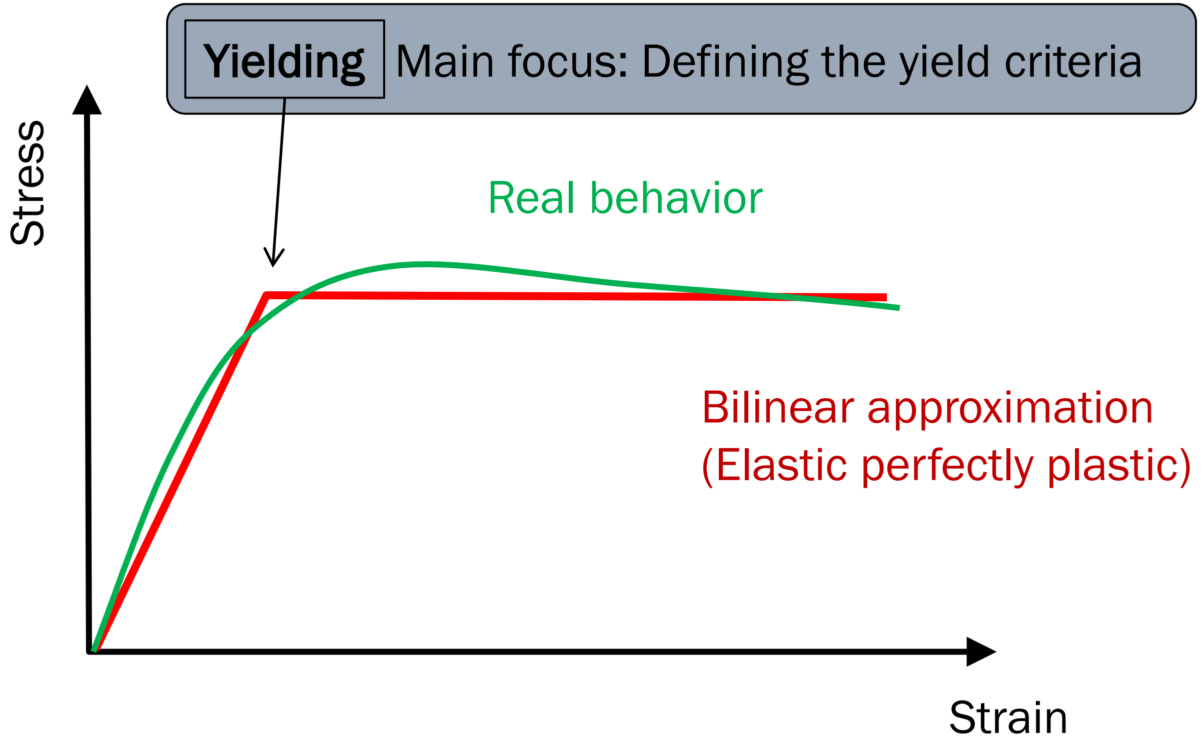
DRUCKER-PRAGER

MOHR-COULOMB

Elastic-perfectly plastic model

Elastic-perfectly plastic is a bilinear approximation defined by yield limit:

(Perfect plasticity: full plasticity without evolution, i.e., Horizontal linear part in stress-strain plot)



- Pre-yield behaviour : Elasticity
- Yield limit – to be defined
- Post-yield behaviour: Perfect plasticity

Invariants of stress tensor

- $\boldsymbol{J_1}$: First invariant of the principal stress tensor σ_{ij} (its trace)

$$J_1 = tr(\sigma_{ij}) = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

- $\boldsymbol{J_{2D}}$: Second invariant of the deviatoric stress tensor s_{ij}

$$J_{2D} = \frac{1}{2}(s_{ij}^2)$$

- $\boldsymbol{J_3}$: Third invariant of the stress tensor σ_{ij} (its determinant)

$$J_3 = det(\sigma_{ij})$$

Stress tensor

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

$$\sigma_{ij} = p\delta_{ij} + \mathrm{s_{ij}}$$

$$p = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3}$$

Isotropic stress tensor

$$p\delta_{ij} = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}$$

Deviatoric stress tensor

$$s_{ij} = \begin{bmatrix} \sigma_{11} - p & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - p & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - p \end{bmatrix}$$

Invariants of stress tensor in principal-stress space

- $\boldsymbol{J_1}$: First invariant of the principal stress tensor σ_{ij} (its trace)

$$J_1 = \sigma_1 + \sigma_2 + \sigma_3$$

- $\boldsymbol{J_{2D}}$: Second invariant of the deviatoric stress tensor s_{ij}

$$J_{2D} = \frac{1}{2} tr\left(s_{ij}^2\right) = \frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

- $\boldsymbol{J_3}$: Third invariant of the stress tensor σ_{ij} (its determinant)

$$J_3 = det\left(\sigma_{ij}\right) = \sigma_1 \sigma_2 \sigma_3$$

Stress tensor in principal-stress space

$$\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

$$\sigma_{ij} = p\delta_{ij} + \mathrm{s_{ij}}$$

$$p = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

Isotropic stress tensor

$$p\delta_{ij} = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}$$

Deviatoric stress tensor

$$s_{ij} = \begin{bmatrix} \sigma_1 - p & 0 & 0 \\ 0 & \sigma_2 - p & 0 \\ 0 & 0 & \sigma_3 - p \end{bmatrix}$$

Invariants of stress tensor in (p,q) triaxial variables

- $\boldsymbol{J_1}$: First invariant of the principal stress tensor σ_{ij} (its trace)

$$J_1 = 3p$$

- $\boldsymbol{J_{2D}}$: Second invariant of the deviatoric stress tensor s_{ij}

$$J_{2D} = \frac{q^2}{3}$$

- $\boldsymbol{J_3}$: Third invariant of the stress tensor σ_{ij} (its determinant)

$$J_3 = det(\sigma_{ij}) = \left(p + \frac{2}{3}q\right)\left(p - \frac{1}{3}q\right)^2$$

Stress tensor in (p,q) triaxial variables

$$\sigma_{ij} = \begin{bmatrix} p + \frac{2}{3}q & 0 & 0 \\ 0 & p - \frac{1}{3}q & 0 \\ 0 & 0 & p - \frac{1}{3}q \end{bmatrix}$$

$$s_{ij} = \begin{bmatrix} \frac{2}{3}q & 0 & 0 \\ 0 & -\frac{1}{3}q & 0 \\ 0 & 0 & -\frac{1}{3}q \end{bmatrix}$$

Stress tensor in (σ_a, σ_r) triaxial conditions

$$\sigma_{ij} = \begin{bmatrix} \sigma_a & 0 & 0 \\ 0 & \sigma_r & 0 \\ 0 & 0 & \sigma_r \end{bmatrix}$$

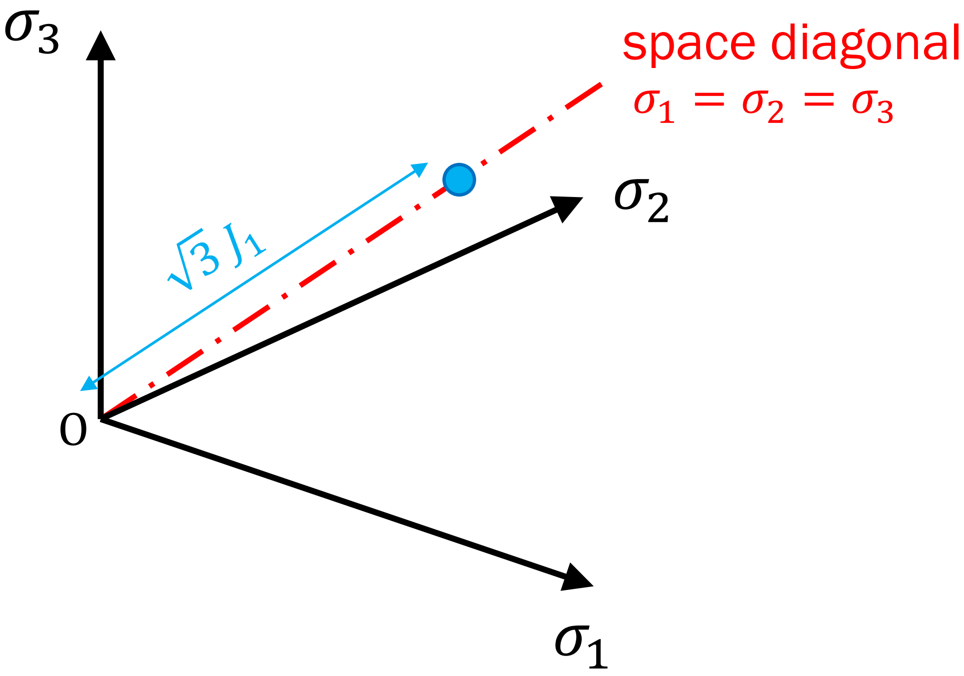
$$p = \sigma_a + 2\sigma_r$$

$$q = \sigma_a - \sigma_r$$

$$\sigma_a = p + \frac{2}{3}q; \quad \sigma_r = p - \frac{1}{3}q$$

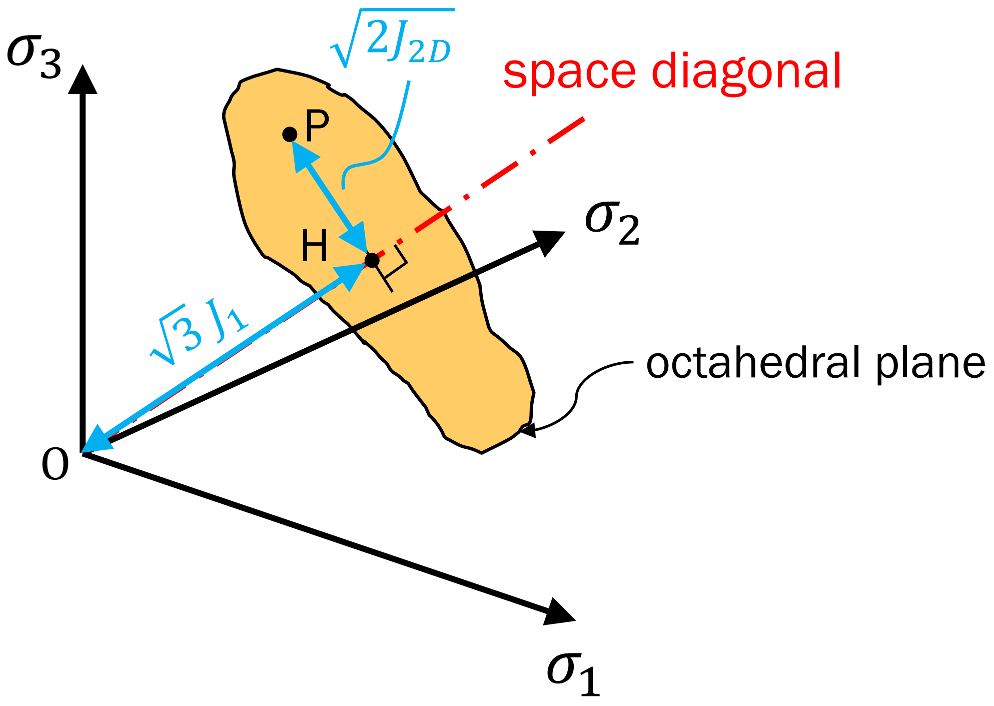
Haigh-Westergaard stress space

- A three-dimensional space where the principal directions have been selected as the coordinate axes



- Along the space diagonal $\sigma_1 = \sigma_2 = \sigma_3 = \frac{J_1}{3}$
- The distance along the space diagonal is $\sqrt{3}\,J_1$

Haigh-Westergaard stress space
Octahedral plane



- It can be seen that:
  - on these planes the mean stress p is constant
  - and it is proportional to the distance from the origin O
- We will use these planes to define the failure criteria in perfect-plasticity models.

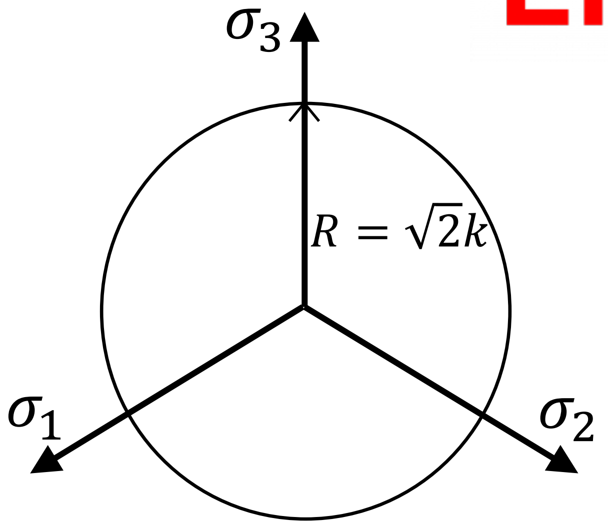
Von Mises criterion

- Initially developed for metals
- Yield function - Independent of spheric stress

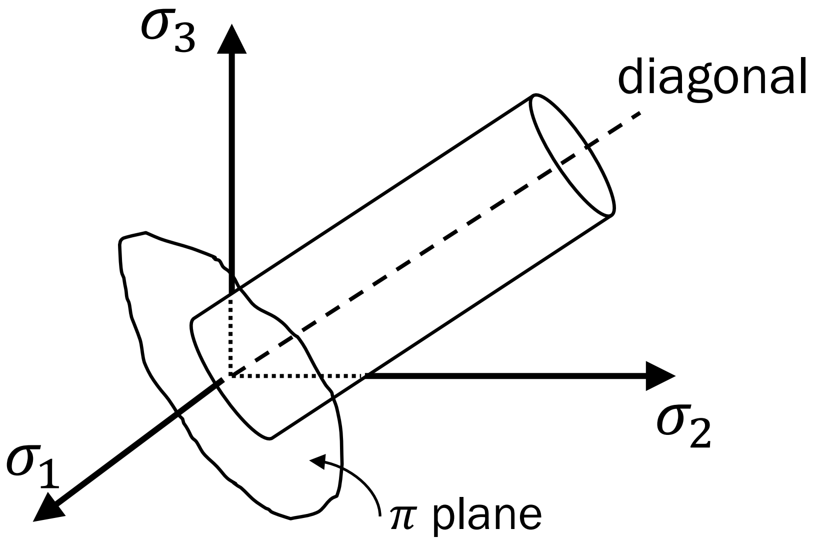
$$F = J_{2D} - k^2$$

Material parameter

$$F = \frac{1}{6} \cdot [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] - k^2 = 0$$



Π plane



Haigh Westergaard space

Von Mises criterion: Triaxial CTC test

$$\frac{1}{6}\cdot\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]-k^2=0$$

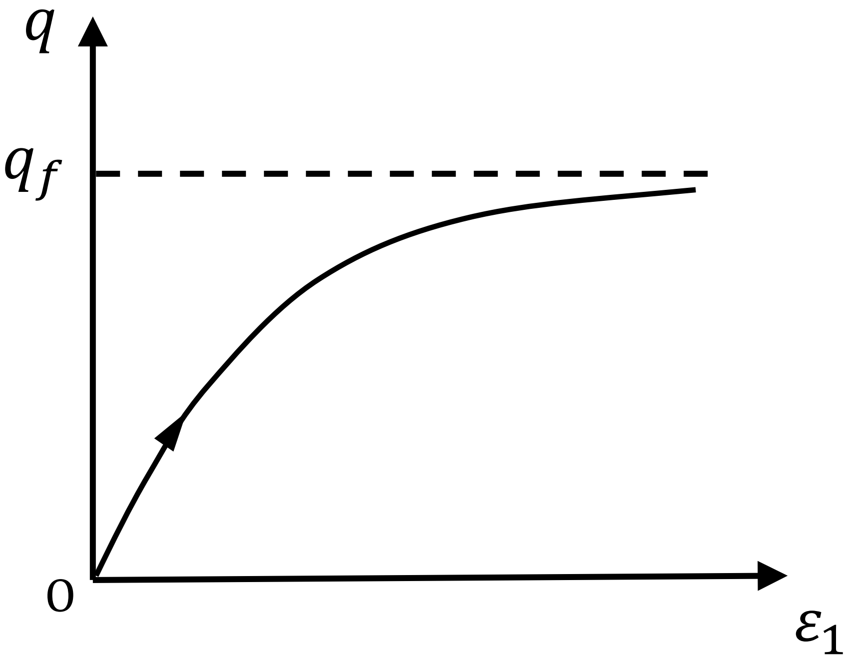
Triaxial CTC test: $\sigma_2 = \sigma_3$

$$\frac{1}{6}\cdot\left[2\cdot(\sigma_1-\sigma_3)^2\right]-k^2=0$$

$$\frac{1}{3}\cdot q_f^2-k^2=0$$

$$k=\frac{1}{\sqrt{3}}\cdot q_f$$

Material parameter

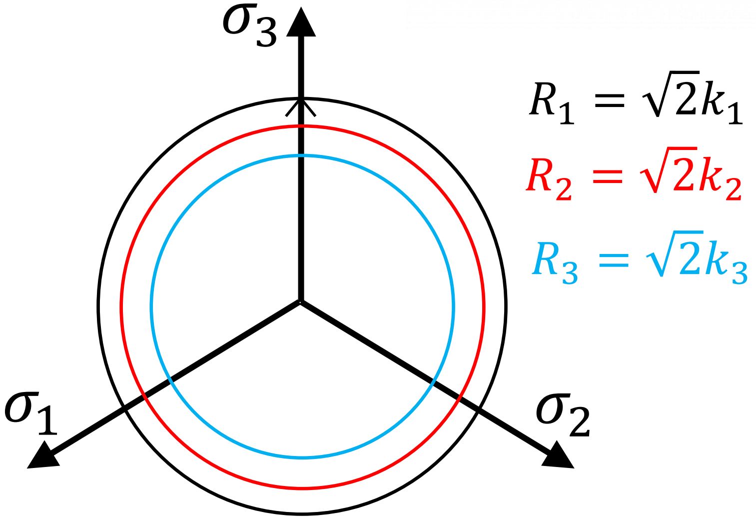


Von Mises criterion: Triaxial CTC test

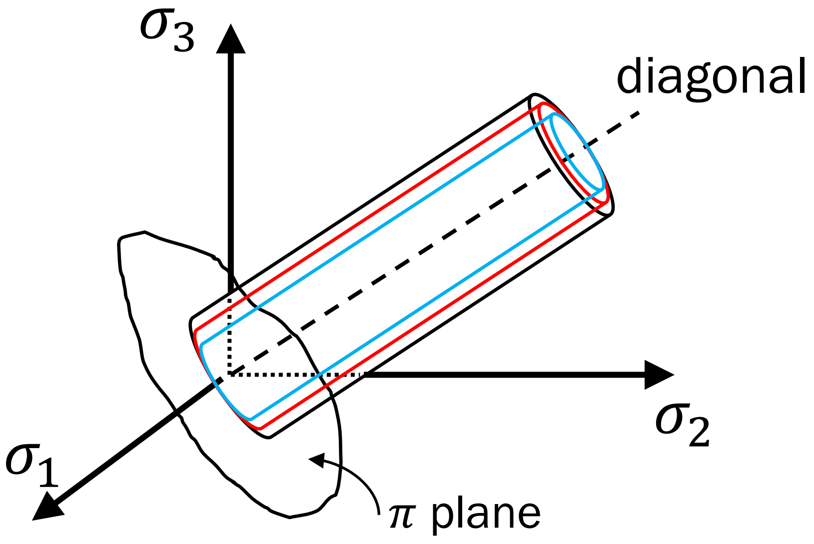
Limitations of the Von Mises criterion for geomaterials:

$$\boxed{k = \frac{1}{\sqrt{3}} \cdot q_f}$$

- Three CTC tests carried out at different initial mean effective stress p' will result in three different q_f and three different values of k
- In other words, the Von Mises criterion cannot reproduce the stress-dependent shear strength typical of geomaterials
- The only case when the strength is considered as independent from the confining stress is when the undrained cohesion is used.



Π plane



Haigh Westergaard space

Von Mises criterion: link to classical soil mechanics

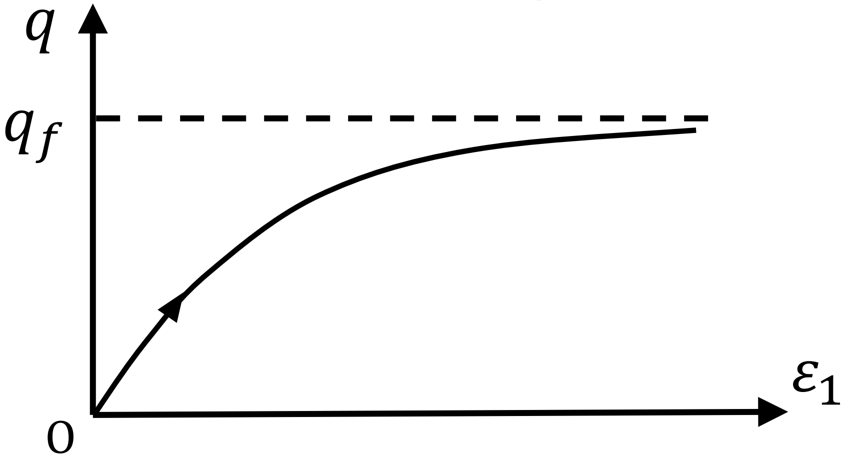
Undrained- Unconsolidated triaxial test (UU) on saturated clay

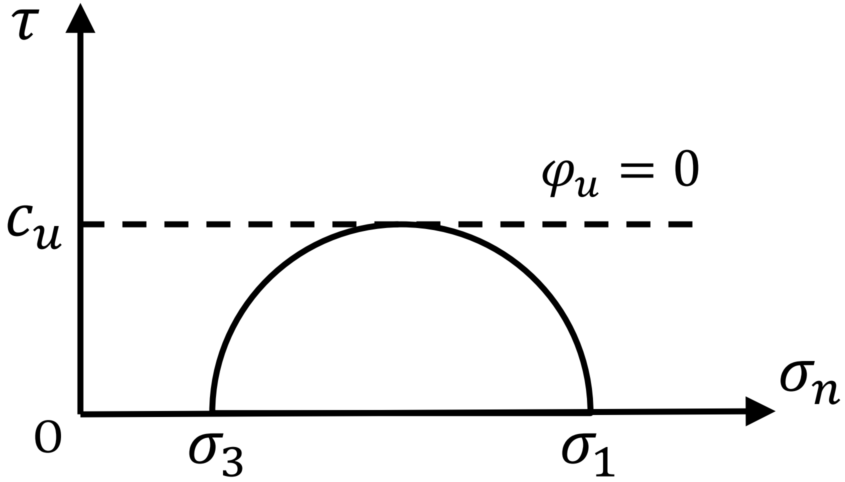
$$c_u = \tau_{\max} = \frac{1}{2} \cdot (\sigma_1 - \sigma_3)_f = \frac{1}{2} \cdot q_f$$

Undrained shear strength

$$\boxed{k = \frac{2}{\sqrt{3}} \cdot c_u}$$

Material parameter





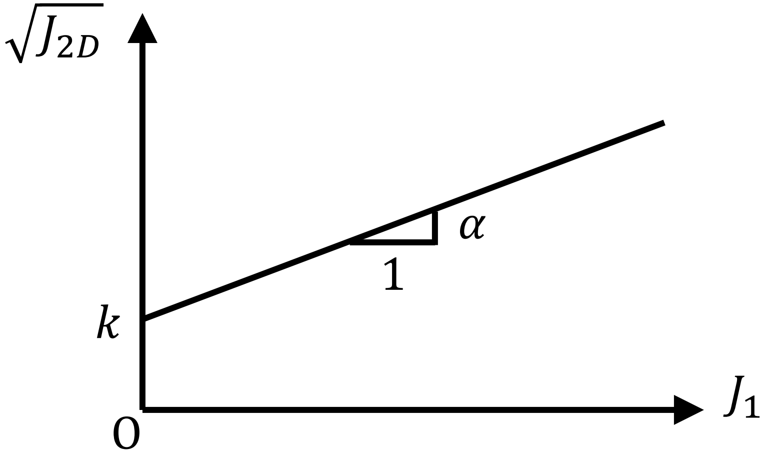
Drucker-Prager criterion

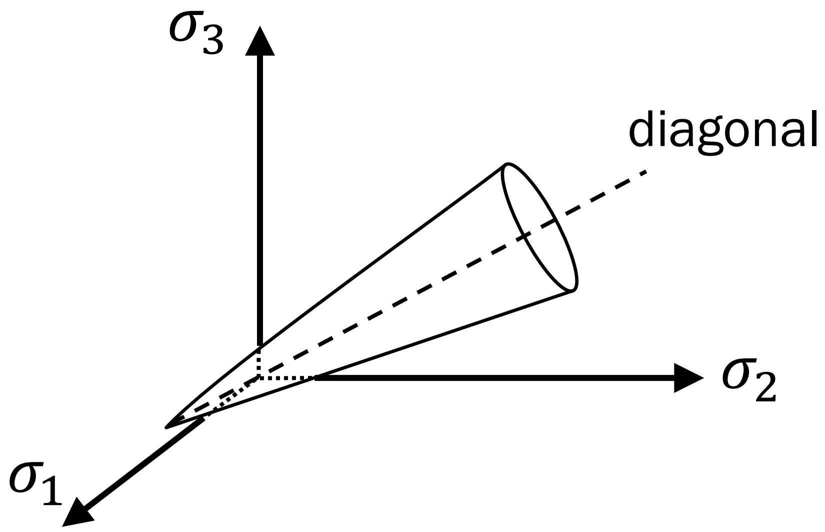
- Improvement of Von Mises for geomaterials
- Yield function - Dependent on spheric stress

Material parameters

$$F = \sqrt{J_{2D}} - \alpha \cdot J_1 - k$$

$$\sqrt{\frac{1}{6}\cdot\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]} - \alpha\cdot(\sigma_1+\sigma_2+\sigma_3) - k = 0$$





Haigh-Westergaard space

Drucker-Prager criterion: Triaxial CTC test

$$\sqrt{\frac{1}{6}\cdot\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]}-\alpha\cdot(\sigma_1+\sigma_2+\sigma_3)-k=0$$

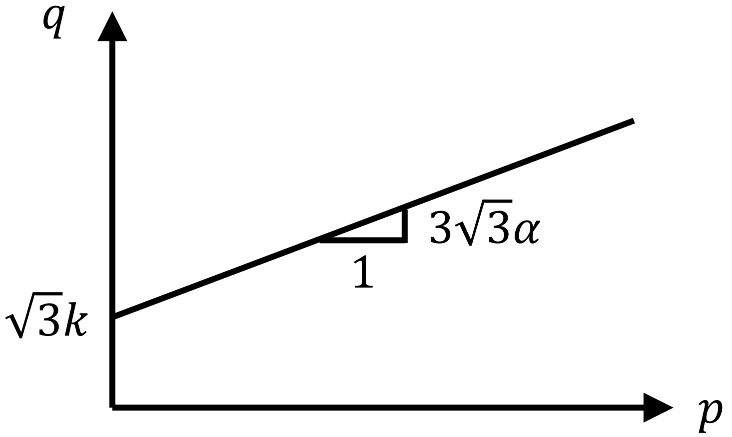
Triaxial CTC test: $\sigma_2 = \sigma_3$

$$\frac{1}{\sqrt{3}}\cdot(\sigma_1-\sigma_3)-\alpha\cdot(\sigma_1+2\cdot\sigma_3)-k=0$$

$$\frac{1}{\sqrt{3}}\cdot q-3\cdot\alpha\cdot p-k=0$$

$$\boxed{q=3\cdot\sqrt{3}\cdot\alpha\cdot p+\sqrt{3}\cdot k}$$

Material parameters



Drucker-Prager criterion: link to classical soil mechanics

Drucker-Prager parameters can be adjusted to the soil mechanics parameters “Cohesion” and “shear strength angle” on Mohr circle plane

$$\boxed{\alpha = \frac{2 \cdot \sin \varphi'}{\sqrt{3} \cdot (3 - \sin \varphi')}}$$

$$\boxed{k = \frac{6 \cdot c' \cdot \cos \varphi'}{\sqrt{3} \cdot (3 - \sin \varphi')}}$$

Mohr-Coulomb criterion: soil mechanics

- Developed for geomaterials
- Mohr-Coulomb shear strength in soil mechanics

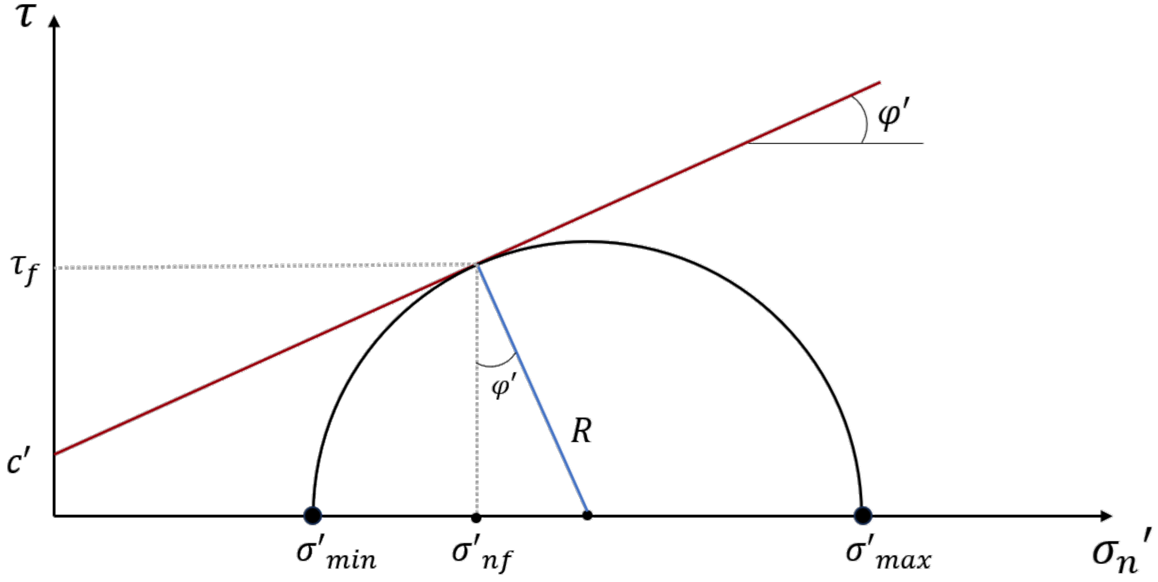
Max. shear stress at failure

Normal effective stress

$$\tau_f = c' + \sigma'_{nf} \tan \varphi'$$

Material parameter: Intercept Cohesion

Material parameter: Shear strength angle



Mohr-Coulomb criterion: soil mechanics

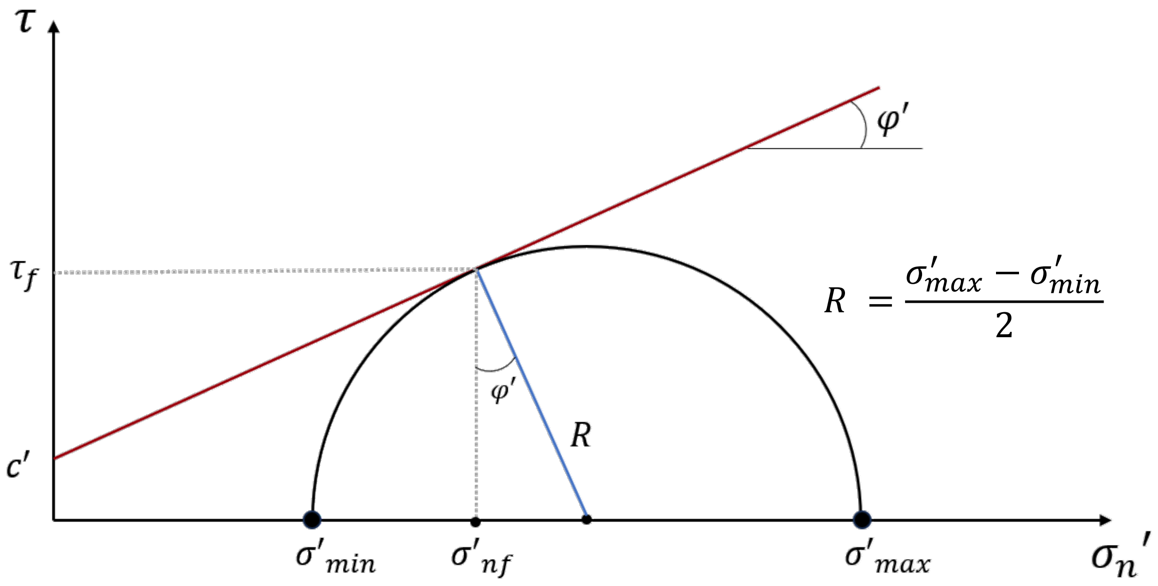
$$\sigma'_{nf} = \frac{\sigma'_{max} + \sigma'_{min}}{2} - R\sin\varphi' = \frac{\sigma'_{max} + \sigma'_{min}}{2} - \frac{\sigma'_{max} - \sigma'_{min}}{2}\sin\varphi'$$

$$\tau_f = R\cos\varphi' = \frac{\sigma'_{max} - \sigma'_{min}}{2}\cos\varphi'$$

$$\tau_f = c' + \sigma'_{nf}\tan\varphi'$$

$$\downarrow$$

$$\frac{\sigma'_{max} - \sigma'_{min}}{2} = \frac{\sigma'_{max} + \sigma'_{min}}{2}\sin\varphi' + c'\cos\varphi'$$



$$F = -\frac{\sigma'_{max} - \sigma'_{min}}{2} + \frac{\sigma'_{max} + \sigma'_{min}}{2}\sin\varphi' + c'\cos\varphi' = 0$$

Yield function

Mohr-Coulomb criterion: Triaxial formulation

$$\boxed{F = -\frac{\sigma'_{max} - \sigma'_{min}}{2} + \frac{\sigma'_{max} + \sigma'_{min}}{2} \sin\varphi' + c' \cos\varphi' = 0}$$

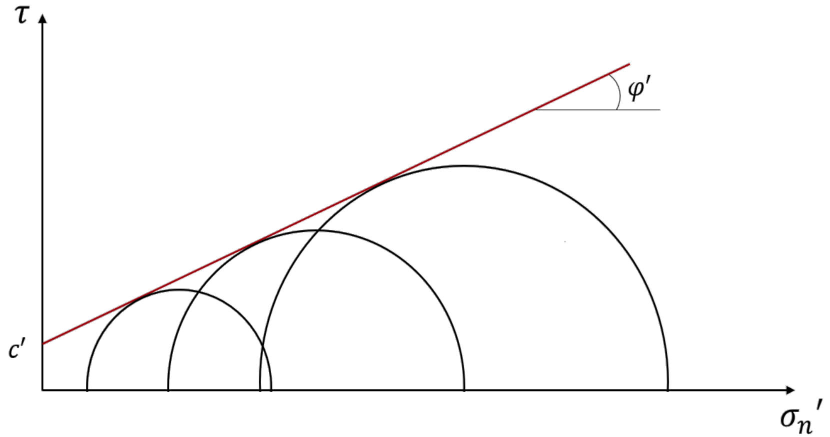
In triaxial compression test:

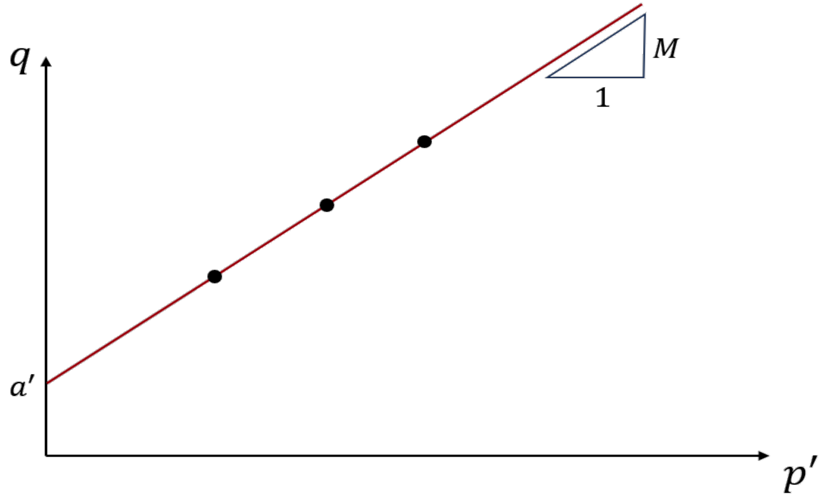
$$q = \sigma'_{max} - \sigma'_{min} \qquad p' = \frac{\sigma'_{max} + 2\sigma'_{min}}{3} \longrightarrow \sigma'_{max} = p' + \frac{2}{3}q \qquad \sigma'_{min} = p' - \frac{q}{3}$$

$$\boxed{q_f = \frac{6\sin\varphi'}{3 - \sin\varphi'} p'_f + \frac{6\cos\varphi'}{3 - \sin\varphi'} c' = M p'_f + a'}$$

With:

$$M = \frac{6\sin\varphi'}{3 - \sin\varphi'} \qquad a' = \frac{6\cos\varphi'}{3 - \sin\varphi'} c'$$





Mohr-Coulomb criterion: Triaxial CTC test

Yield function

$$F = -\frac{\sigma_a' - \sigma_r'}{2} + \frac{\sigma_a' + \sigma_r'}{2} \sin\varphi' + c' \cos\varphi' = 0$$

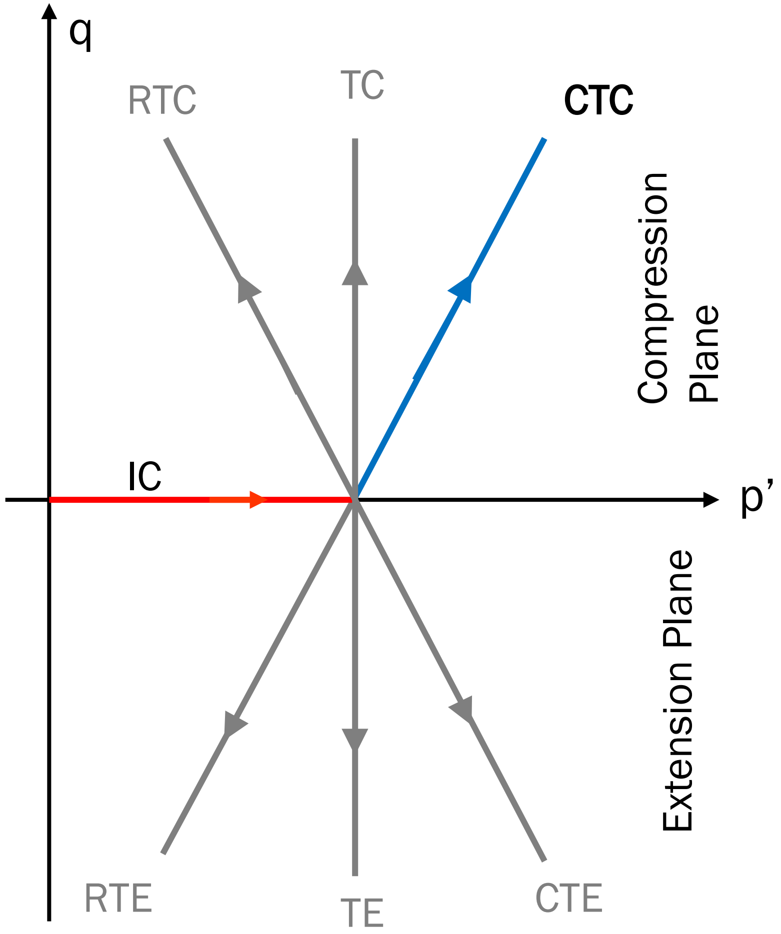
Stress condition

$$q = \sigma_a' - \sigma_r'$$

$$p' = \frac{1}{3}(\sigma_a' + 2\sigma_r')$$

Yield function in terms of triaxial stress variables:

$$q_f = \frac{6 \sin\varphi'}{3 - \sin\varphi'} p_f' + \frac{6 \cos\varphi'}{3 - \sin\varphi'} c' = M p_f' + a'$$



Mohr-Coulomb criterion: Triaxial RTE test

Yield function

$$F = -\frac{\sigma_r' - \sigma_a'}{2} + \frac{\sigma_r' + \sigma_a'}{2}\sin\varphi' + c'\cos\varphi' = 0$$

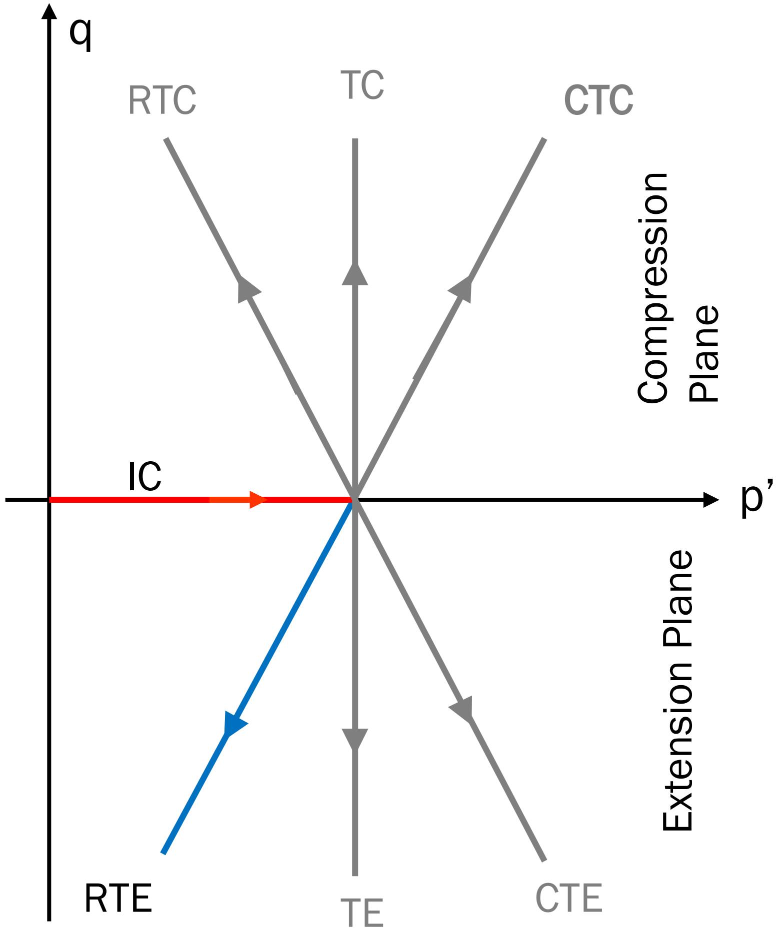
Stress condition

$$q = \sigma_r' - \sigma_a'$$

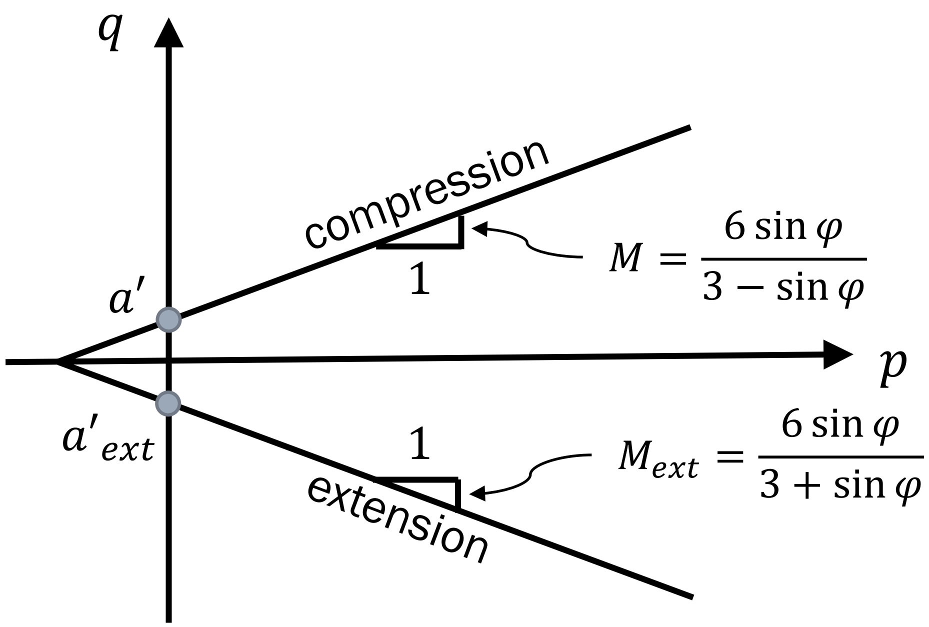
$$p' = \frac{1}{3}(2\sigma_r' + \sigma_a')$$

Yield function in terms of triaxial stress variables:

$$q_f = -\frac{6\sin\varphi'}{3+\sin\varphi'}p_f' - \frac{6\cos\varphi'}{3+\sin\varphi'}c' = -M_{ext}p_f' - a_{ext}'$$



Mohr-Coulomb criterion: Triaxial envelope



$M = \dfrac{6 \sin\varphi}{3 - \sin\varphi}$

$M_{ext} = \dfrac{6 \sin\varphi}{3 + \sin\varphi}$

CTC

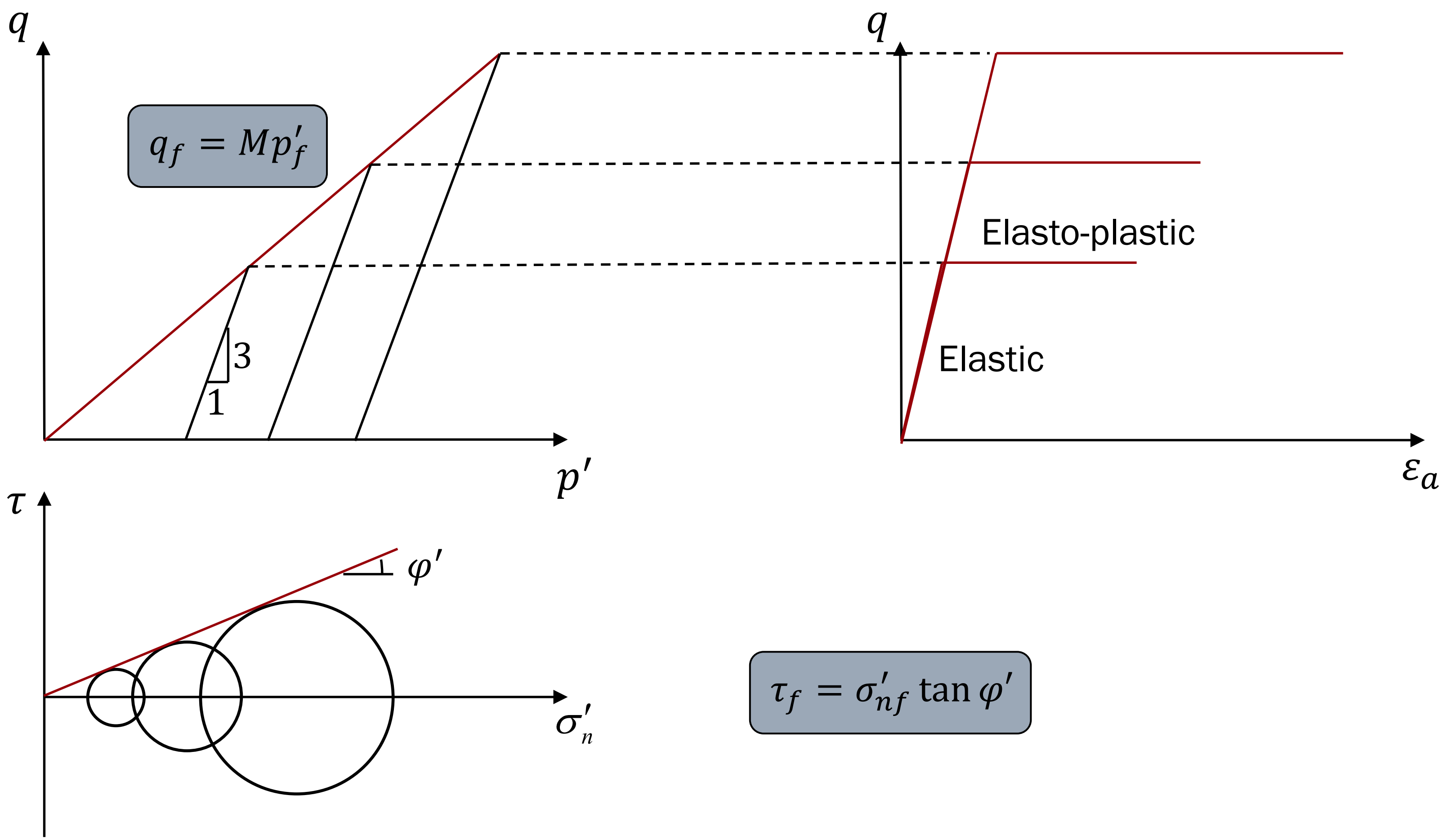
$$q_f = M p'_f + a'$$

RTE

$$q_f = -M_{ext} p'_f - a'_{ext}$$

Mohr-Coulomb criterion: Triaxial envelope

- If there is no cohesion:

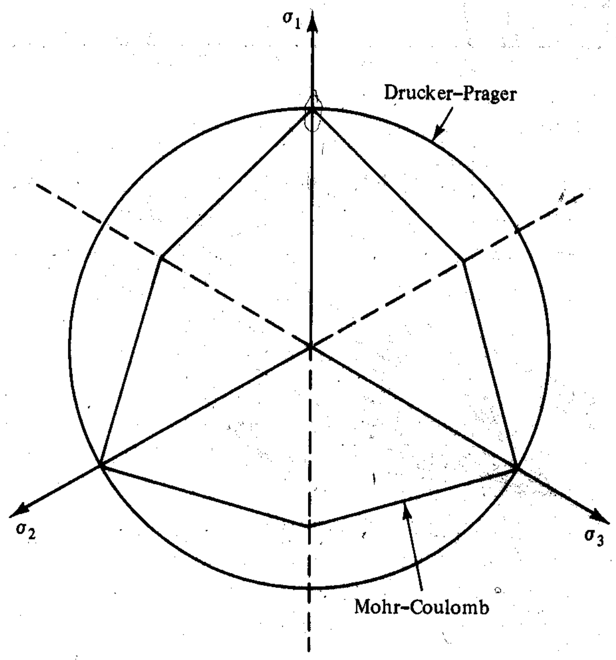


$q_f = M p'_f$

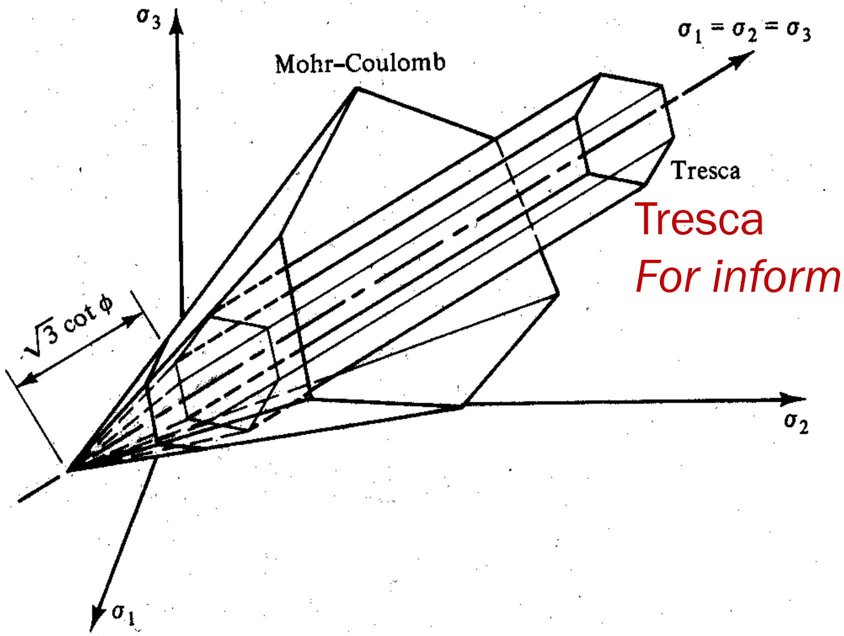
$\tau_f = \sigma'_{nf} \tan \varphi'$

Mohr-Coulomb criterion: Graphical representation

Π plane



Haigh-Westergaard space



Tresca

For information only

For information only: **MC yield criteria in terms of stress invariants**

$$F = J_1 \cdot \sin\phi + \sqrt{J_{2D}} \cdot \cos\theta - \frac{\sqrt{J_{2D}}}{3} \cdot \sin\phi \cdot \sin\theta - c \cdot \cos\phi = 0$$

$$\theta = -\frac{1}{3} \cdot \sin^{-1}\left(-\frac{3 \cdot \sqrt{3}}{2} \cdot \frac{J_{3D}}{J_{2D}^{3/2}}\right)$$

Mohr-Coulomb elasto-perfectly plastic model

- Isotropic linear elasticity

$$\begin{pmatrix} \delta p' \\ \delta q \end{pmatrix} = \begin{bmatrix} K & 0 \\ 0 & 3G \end{bmatrix} \begin{pmatrix} \delta \varepsilon^e_{vol} \\ \delta \varepsilon^e_q \end{pmatrix}$$

- Yield function

$$F(\sigma_i, p_k)$$

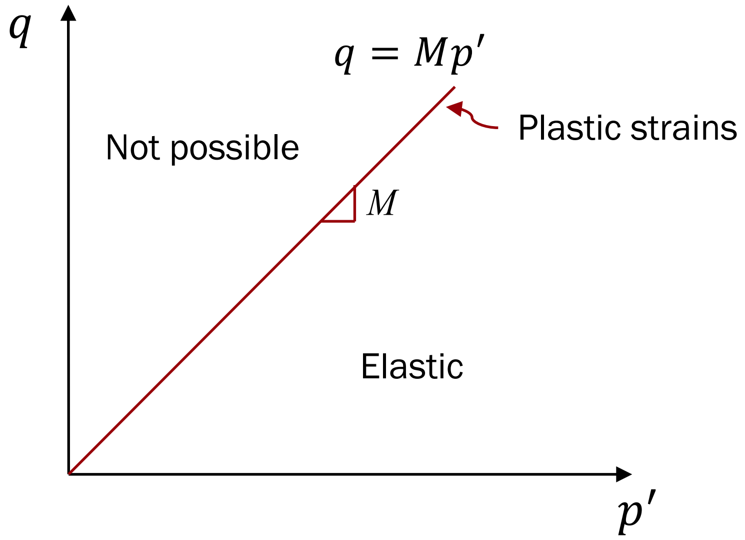
$$F(p', q, p_k) = q - Mp'$$

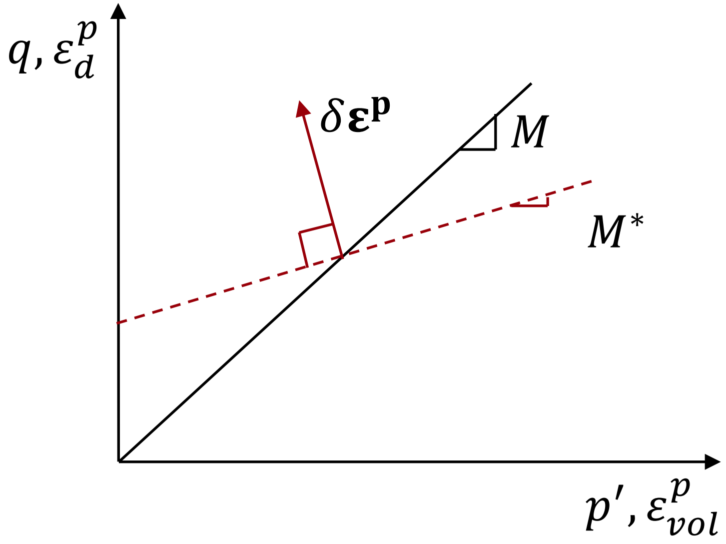
One geometrical parameter: $p_k = M$

- Plastic potential

$$g(\sigma_i, r_k)$$

$$g(p', q, r_k) = q - M^* p' + k$$



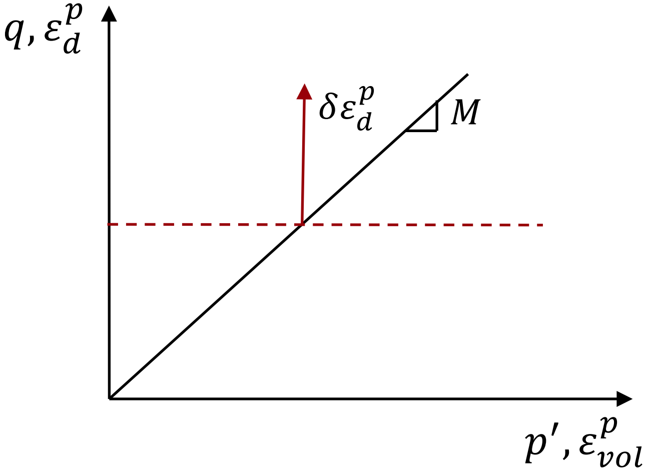


Mohr-Coulomb elasto-perfectly plastic model

$$\delta\varepsilon_i^p = \lambda \frac{\partial g}{\partial \sigma_i}; \qquad \begin{pmatrix} \delta\varepsilon_{vol}^p \\ \delta\varepsilon_d^p \end{pmatrix} = \lambda \begin{pmatrix} \dfrac{\partial g}{\partial p'} \\ \dfrac{\partial g}{\partial q} \end{pmatrix} = \lambda \begin{pmatrix} -M^* \\ 1 \end{pmatrix}$$

$$\frac{\delta\mathcal{E}_{vol}^p}{\delta\mathcal{E}_d^p} = -M^*$$

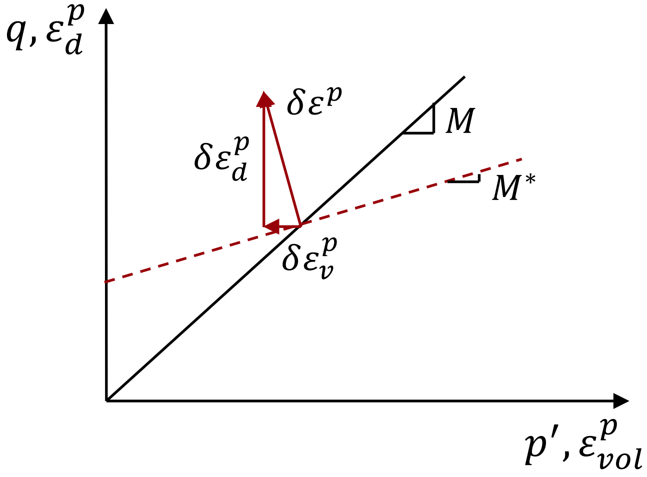
If $M^* = 0$



$\delta\varepsilon_v^p = 0$ No plastic volume deformation during shearing

NO DILATANCY/NO COMPACTION

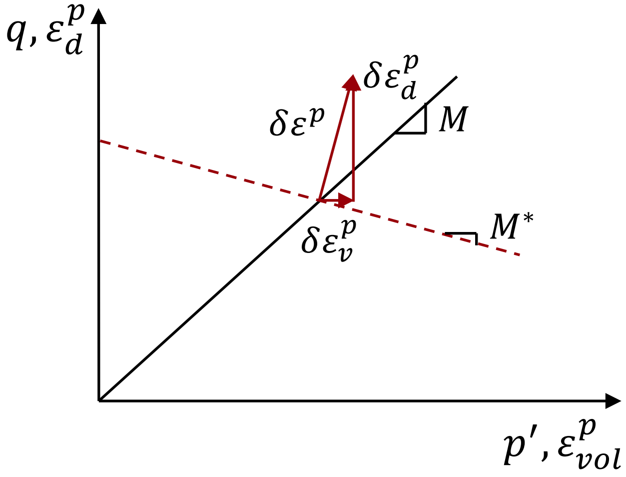
If $M^* > 0$



$\delta\varepsilon_v^p < 0$

DILATANCY

If $M^* < 0$



$\delta\varepsilon_v^p > 0$

COMPACTION

Perfect plasticity

- For perfect plasticity, F is fixed by definition

$$\delta p_k = 0 \;\rightarrow dF = \frac{\partial F}{\partial \sigma_i}\delta\sigma_i = 0$$

- Solution of the elasto-plastic problem for perfect plasticity

$$\delta\sigma_i = D_{ij}^e \delta\varepsilon_j^e = D_{ij}^e\left(\delta\varepsilon_j - \delta\varepsilon_j^p\right) = D_{ij}^e\delta\varepsilon_j - \mu D_{ij}^e\frac{\partial g}{\partial \sigma_j}$$

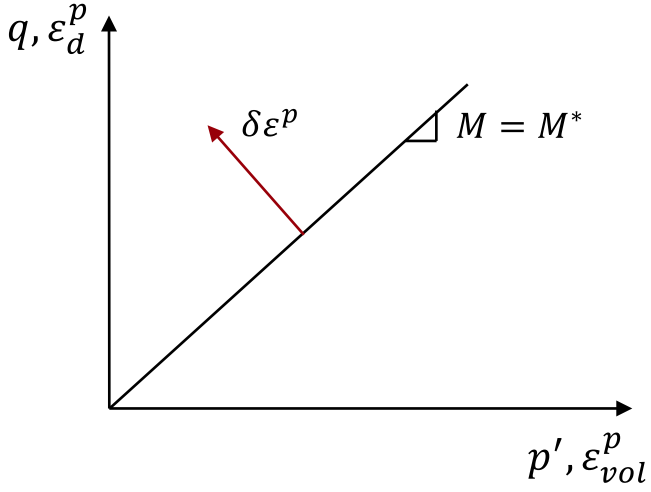
- Consistency equation

$$\frac{\partial F}{\partial \sigma_i}\left(D_{ij}^e\delta\varepsilon_j - \mu D_{ij}^e\frac{\partial g}{\partial \sigma_j}\right) = 0 \rightarrow \mu = \frac{\frac{\partial F}{\partial \sigma_i}D_{ij}^e\delta\varepsilon_j}{\frac{\partial F}{\partial \sigma_i}D_{ij}^e\frac{\partial g}{\partial \sigma_j}}$$

$$\boxed{\delta\sigma_i = \left(D_{ij}^e - \frac{D_{ij}^e\frac{\partial g}{\partial \sigma_j}\frac{\partial F}{\partial \sigma_i}D_{ij}^e}{\frac{\partial F}{\partial \sigma_i}D_{ij}^e\frac{\partial g}{\partial \sigma_j}}\right)\delta\varepsilon_j}$$

Mohr-Coulomb elasto-perfectly plastic model

If an associated flow rule is adopted: $\boxed{M^* = M}$ ⇨



No energy dissipation

Combining all the basic ingredients:

$$\delta\sigma_i' = \left(D_{ij}^e - \frac{D_{ij}^e \frac{\partial g}{\partial \sigma_j} \frac{\partial F}{\partial \sigma_i} D_{ij}^e}{\frac{\partial F}{\partial \sigma_i} D_{ij}^e \frac{\partial g}{\partial \sigma_j}} \right) \delta\varepsilon_j$$

- $\delta\sigma_i' = \begin{pmatrix} \delta p' \\ \delta q \end{pmatrix}$
- $\delta\varepsilon_j = \begin{pmatrix} \delta\varepsilon_{vol} \\ \delta\varepsilon_q \end{pmatrix}$
- $D_{ij}^e = \begin{bmatrix} K & 0 \\ 0 & 3G \end{bmatrix}$
- $\frac{\partial g}{\partial \sigma_j} = \begin{pmatrix} \frac{\partial g}{\partial p'} \\ \frac{\partial g}{\partial q} \end{pmatrix} = \begin{pmatrix} -M^* \\ 1 \end{pmatrix}$
- $\frac{\partial F}{\partial \sigma_i} = \begin{pmatrix} \frac{\partial F}{\partial p'} \\ \frac{\partial F}{\partial q} \end{pmatrix} = \begin{pmatrix} -M \\ 1 \end{pmatrix}$

Mohr-Coulomb elasto-perfectly plastic model

$$\rightarrow \begin{bmatrix} K & 0 \\ 0 & 3G \end{bmatrix} - \frac{\begin{bmatrix} K & 0 \\ 0 & 3G \end{bmatrix}\begin{pmatrix} -M^* \\ 1 \end{pmatrix}\begin{pmatrix} -M & 1 \end{pmatrix}\begin{bmatrix} K & 0 \\ 0 & 3G \end{bmatrix}}{\begin{pmatrix} -M & 1 \end{pmatrix}\begin{bmatrix} K & 0 \\ 0 & 3G \end{bmatrix}\begin{pmatrix} -M^* \\ 1 \end{pmatrix}} = \begin{bmatrix} K & 0 \\ 0 & 3G \end{bmatrix} - \frac{\begin{pmatrix} -KM^* \\ 3G \end{pmatrix}\begin{pmatrix} -KM & 3G \end{pmatrix}}{\begin{pmatrix} -KM & 3G \end{pmatrix}\begin{pmatrix} -M^* \\ 1 \end{pmatrix}} \rightarrow$$

$$\rightarrow \begin{bmatrix} K & 0 \\ 0 & 3G \end{bmatrix} - \frac{\begin{bmatrix} K^2M^*M & -3KM^*G \\ -3KGM & 9G^2 \end{bmatrix}}{MKM^* + 3G} = \begin{bmatrix} K & 0 \\ 0 & 3G \end{bmatrix} - \frac{1}{MKM^* + 3G}\begin{bmatrix} K^2M^*M & -3KM^*G \\ -3KGM & 9G^2 \end{bmatrix} \rightarrow$$

$$\rightarrow \begin{bmatrix} \cancel{K^2MM^*} + 3GK - \cancel{K^2M^*M} & +3KM^*G \\ +3KMG & 3GKMM^* + \cancel{9G^2} - \cancel{9G^2} \end{bmatrix} \frac{1}{MKM^* + 3G} = \frac{3GK}{MKM^* + 3G}\begin{bmatrix} 1 & M^* \\ M & MM^* \end{bmatrix}$$

$$\boxed{\begin{pmatrix} \delta p' \\ \delta q \end{pmatrix} = \frac{3GK}{MKM^* + 3G}\begin{bmatrix} 1 & M^* \\ M & MM^* \end{bmatrix}\begin{pmatrix} \delta\varepsilon_v \\ \delta\varepsilon_d \end{pmatrix}}$$

Conclusions

Conclusions

- Non-linear behaviour of geomaterials can be described by plasticity.

- An essential element of plasticity is the yield criteria which theoretically separates the Elastic and Plastic domain of behaviour

- Perfect plasticity is a rather simple approach, initially developed for metals, to model the elasto-plastic behaviour of geomaterials

- Elastic-perfectly plastic models can be successfully used to model the mechanical behaviour of geomaterials in different applications, provided that the limitations and simplifying hypotheses are appropriately taken into account.

Thank you for your attention